Optimising Multigrid Solvers Using Grammar-Guided Genetic Programming

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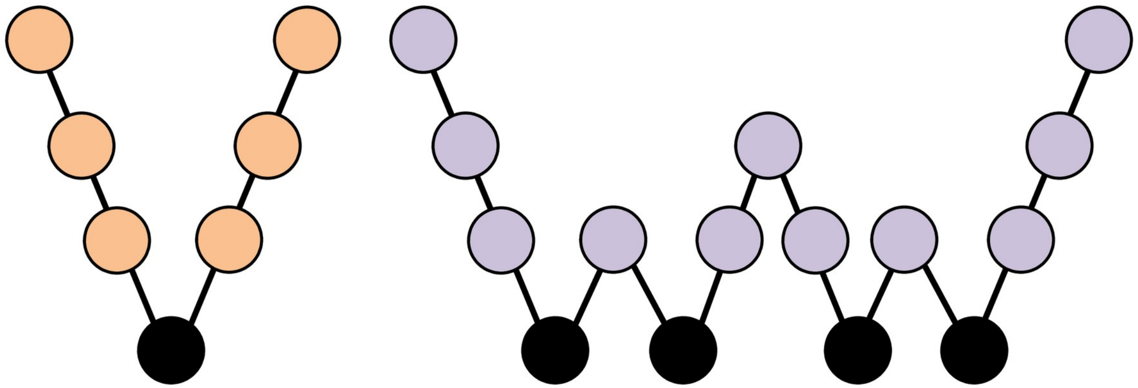
Introduction

$$Ax = b$$

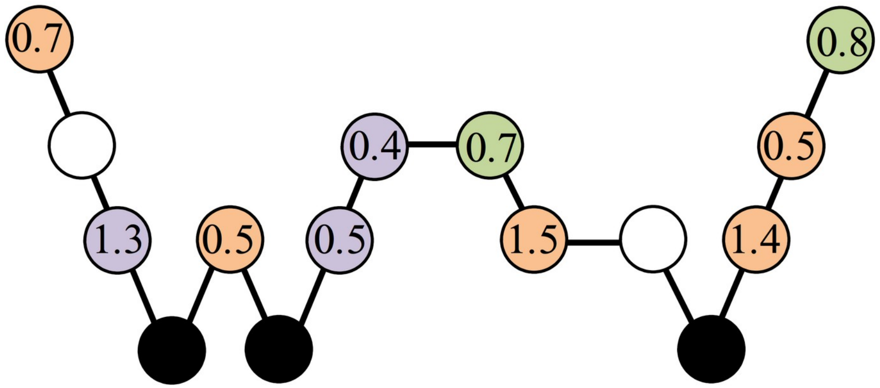
Linear systems of equations arising from many PDEs can be efficiently solved using multigrid methods.

Motivation



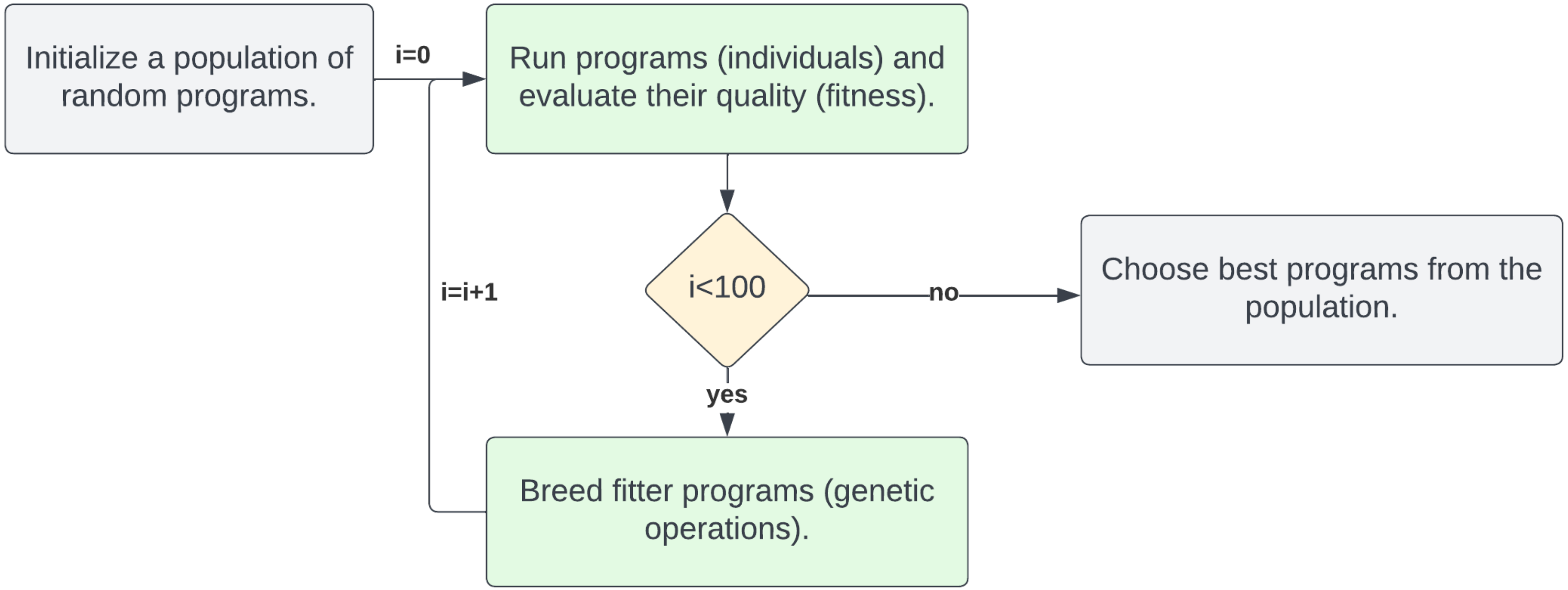
Standard Multigrid Cycles

Large search space !!



Flexible Multigrid Cycles

Genetic Programming – Primer

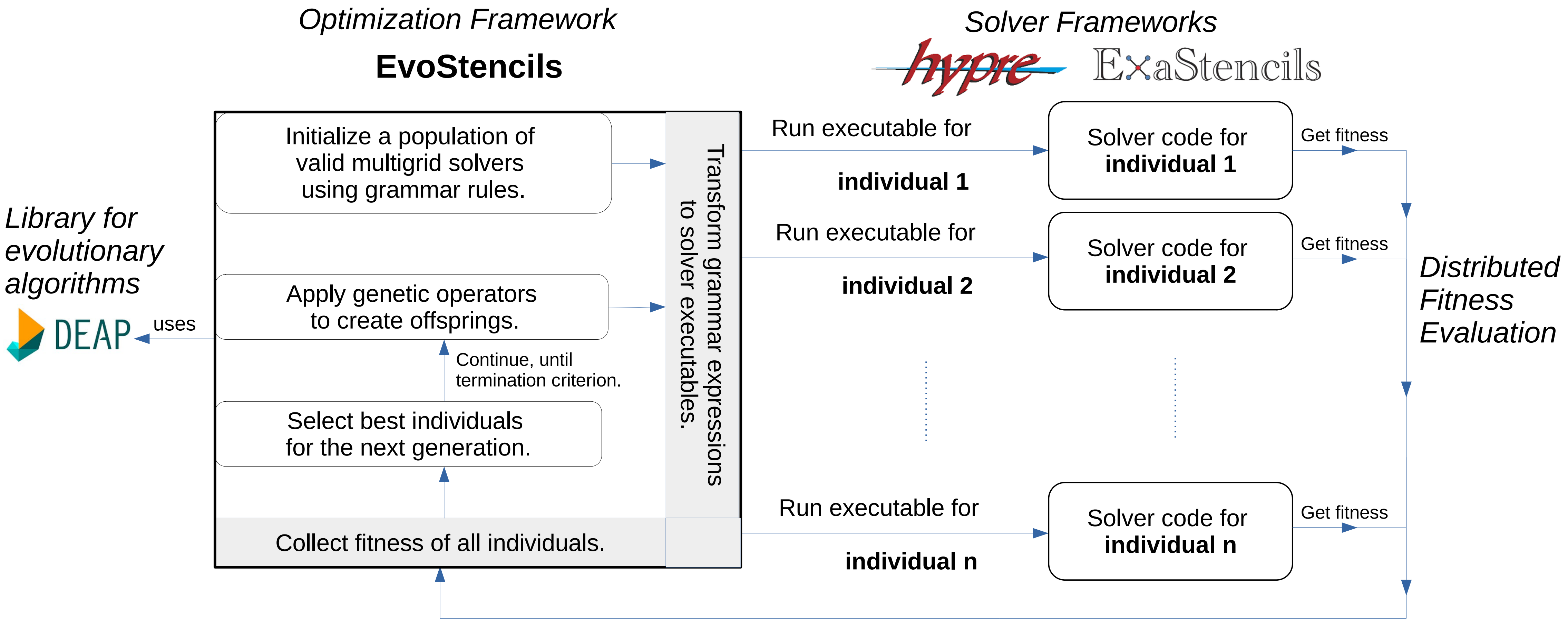


Genetic Programming – Grammar-Guided Approach[1]

$$
\begin{aligned}
\langle S \rangle &\models \langle s_h \rangle \\
\langle s_h \rangle &\models \textsc{iterate}(\langle c_h \rangle,\ \omega,\ \langle \mathcal{P} \rangle) \ \mid\ (u_h^0,\ f_h,\ \lambda,\ \lambda) \\
\langle s_h \rangle &\models \textsc{iterate}(\textsc{apply}(\langle B_h \rangle,\ \langle c_h \rangle),\ \omega,\ \langle \mathcal{P} \rangle) \\
\langle s_h \rangle &\models \textsc{iterate}(\textsc{coarse-grid-correction}(I_{2h}^{h},\ \langle s_{2h} \rangle),\ \omega,\ \langle \mathcal{P} \rangle) \\
\langle c_h \rangle &\models \textsc{residual}(A_h,\ \langle s_h \rangle) \\
\langle B_h \rangle &\models \textsc{inverse}(A_h^{+}) \text{ with } A_h = A_h^{+} + A_h^{-} \\
\langle c_{2h} \rangle &\models \textsc{residual}(A_{2h},\ \langle s_{2h} \rangle) \\
\langle c_{2h} \rangle &\models \textsc{coarse-cycle}(A_{2h},\ u_{2h}^{0},\ \textsc{apply}(I_{h}^{2h},\ \langle c_h \rangle)) \\
\langle s_{2h} \rangle &\models \textsc{iterate}(\langle c_{2h} \rangle,\ \omega,\ \langle \mathcal{P} \rangle) \\
\langle s_{2h} \rangle &\models \textsc{iterate}(\textsc{apply}(\langle B_{2h} \rangle,\ \langle c_{2h} \rangle),\ \omega,\ \langle \mathcal{P} \rangle) \\
\langle s_{2h} \rangle &\models \textsc{iterate}(\textsc{apply}(I_{4h}^{2h},\ \langle c_{4h} \rangle),\ \omega,\ \lambda) \\
\langle B_{2h} \rangle &\models \textsc{inverse}(A_{2h}^{+}) \text{ with } A_{2h} = A_{2h}^{+} + A_{2h}^{-} \\
\langle c_{4h} \rangle &\models \textsc{apply}(A_{4h}^{-1},\ \textsc{apply}(I_{2h}^{4h},\ \langle c_{2h} \rangle)) \\
\langle \mathcal{P} \rangle &\models \textsc{partitioning} \ \mid\ \lambda
\end{aligned}
$$

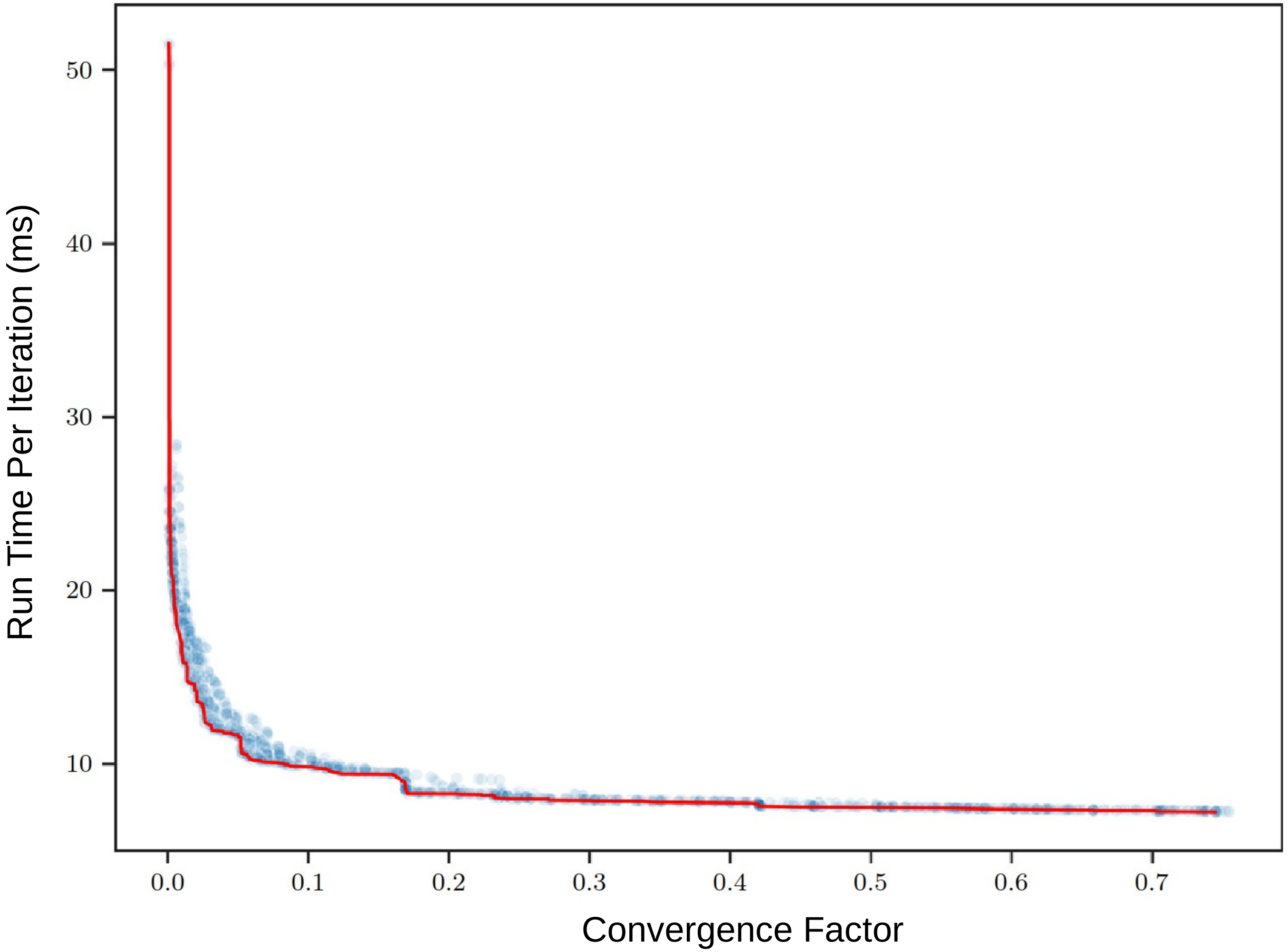
Production rules for constructing three-grid multigrid cycles where each symbol on the left side of the ⊨ sign can be replaced by the corresponding symbol on its right side.

Software Framework



Source code repository: https://github.com/jonas-schmitt/evostencils

Objectives



50
40
30
20
10
Run Time Per Iteration (ms)
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
Convergence Factor

Optimisation Components

Smoothers	Jacobi, Gauss-Seidel, RBGS, Jacobi-Newton, RBGS-Newton…., etc.
Restriction Operator	Full-weighting restriction
Prolongation Operator	Bilinear interpolation
Relaxation factors	(0.1, 0.15, 0.2, …, 1.9)
Coarse grid solver	Gauss Elimination / CG / Jacobi-Newton /…. / etc.

Results – Nonlinear Multigrid[2]

$$-\Delta u(x,y) + \gamma u(x,y) e^{u(x,y)} = f(x,y) \quad \text{in } \Omega,$$

$$u(x,y) = 0 \quad \text{on } \Omega$$

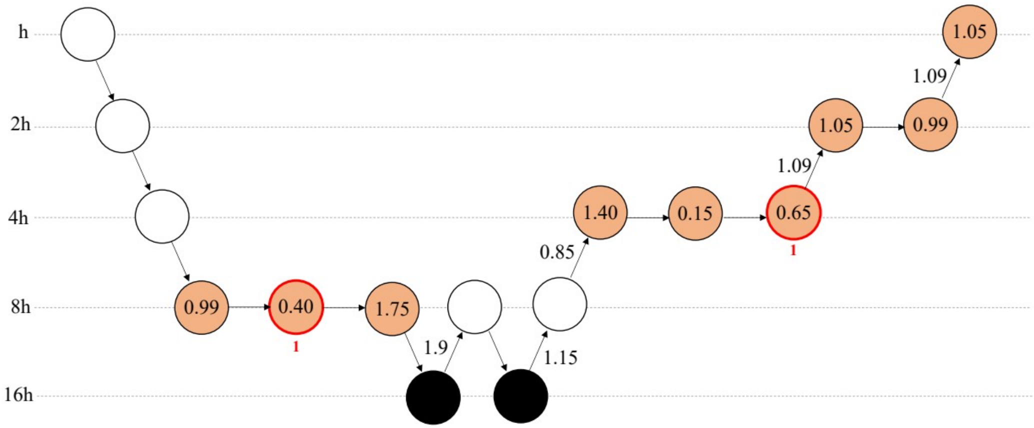
$$f(x,y) = \left((9\pi^2 + \gamma e^{(x^2-x^3) sin(3\pi y)}) \cdot (x^2 - x^3) + 6x - 2 \right) \cdot sin(3\pi y)$$

$$l_{max} = 10 \quad \gamma = 20$$

2D nonlinear problem.

	Runtime (ms)	n_Iterations	Convergence
V(1,1)	306.99	21	0.32
V(1,2)	238.56	14	0.18
V(2,1)	254.78	15	0.21
V(2,2)	253.72	13	0.17
W(2,2)	1017.18	9	0.07
GGGP-1	117.88	7	0.04
GGGP-2	115.09	8	0.05
GGGP-3	118.39	11	0.12

Comparing efficiency of Grammar-Guided Genetic Programming (GGGP / G3P) generated solvers with handcrafted reference methods.



The structure of the GGGP -3 solver . The color of the node denotes the type of operation. Brown: Red-Black Gauss-Seidel smoothing, Black: Coarse Grid Solver, White:No Operation. The borders of the node indicate the type of approximation used in the smoothing operation. Red: Newton's approximation with the number of Newton steps indicated in red text, Black: Picard's approximation. The relaxation factors are indicated inside the node for smoothing and on the edges for coarse-grid correction.

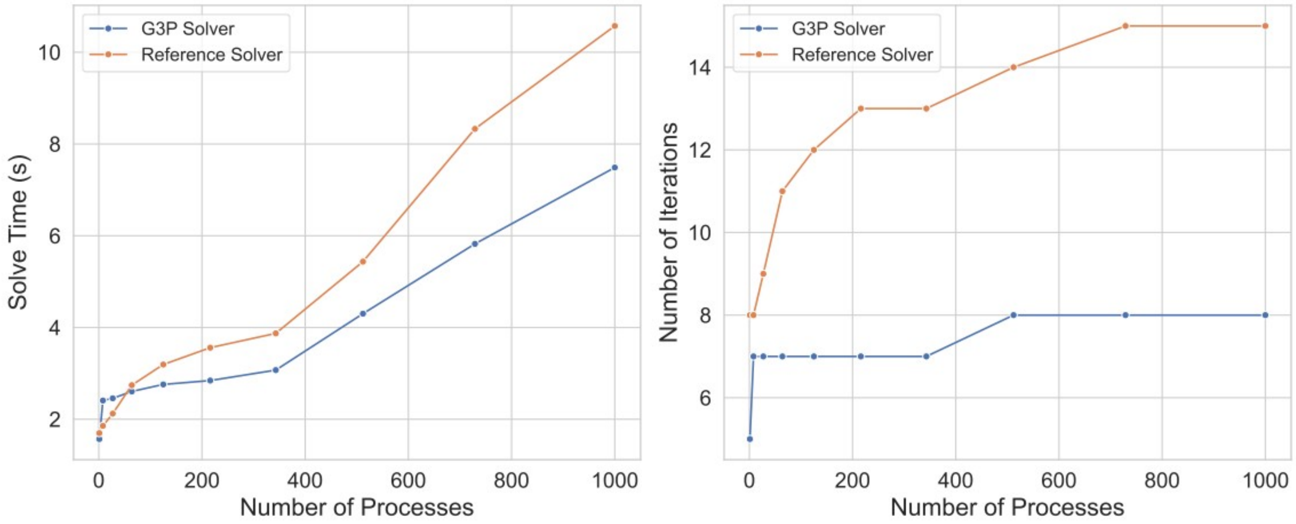
Results – Algebraic Multigrid

$$c_x \frac{\partial^2 \phi}{\partial x^2} + c_y \frac{\partial^2 \phi}{\partial y^2} + c_z \frac{\partial^2 \phi}{\partial z^2} = 0$$

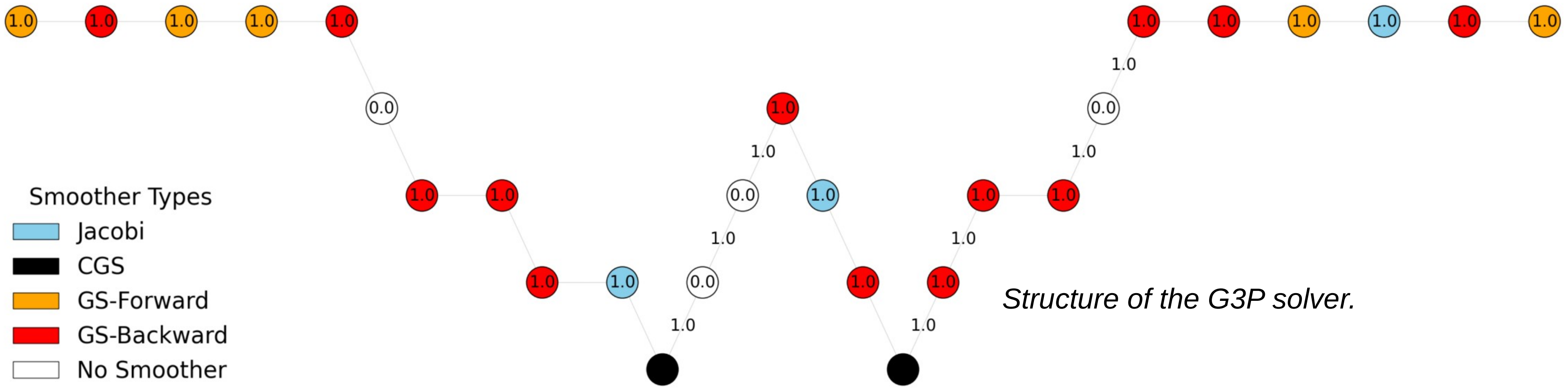
$$c_x = 0.001\,,\quad c_y = 1\,,\quad c_z = 1$$

$$N_x = 100\,,\ N_y = 100\,,\ N_z = 100$$

3D anistropic poisson problem.



Weak scaling of the G3P solver and an optimised V-cycle solver.

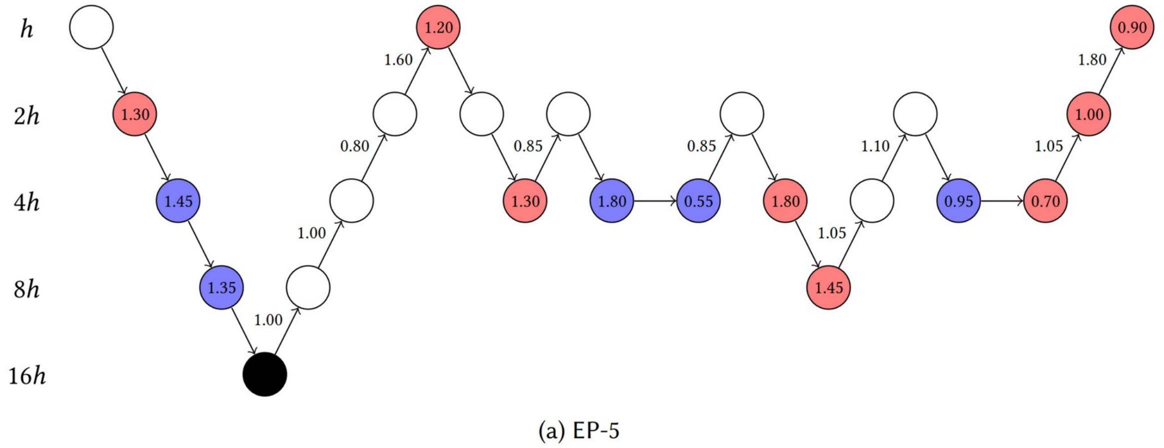


Structure of the G3P solver.

Results – Multigrid Preconditioner [3]

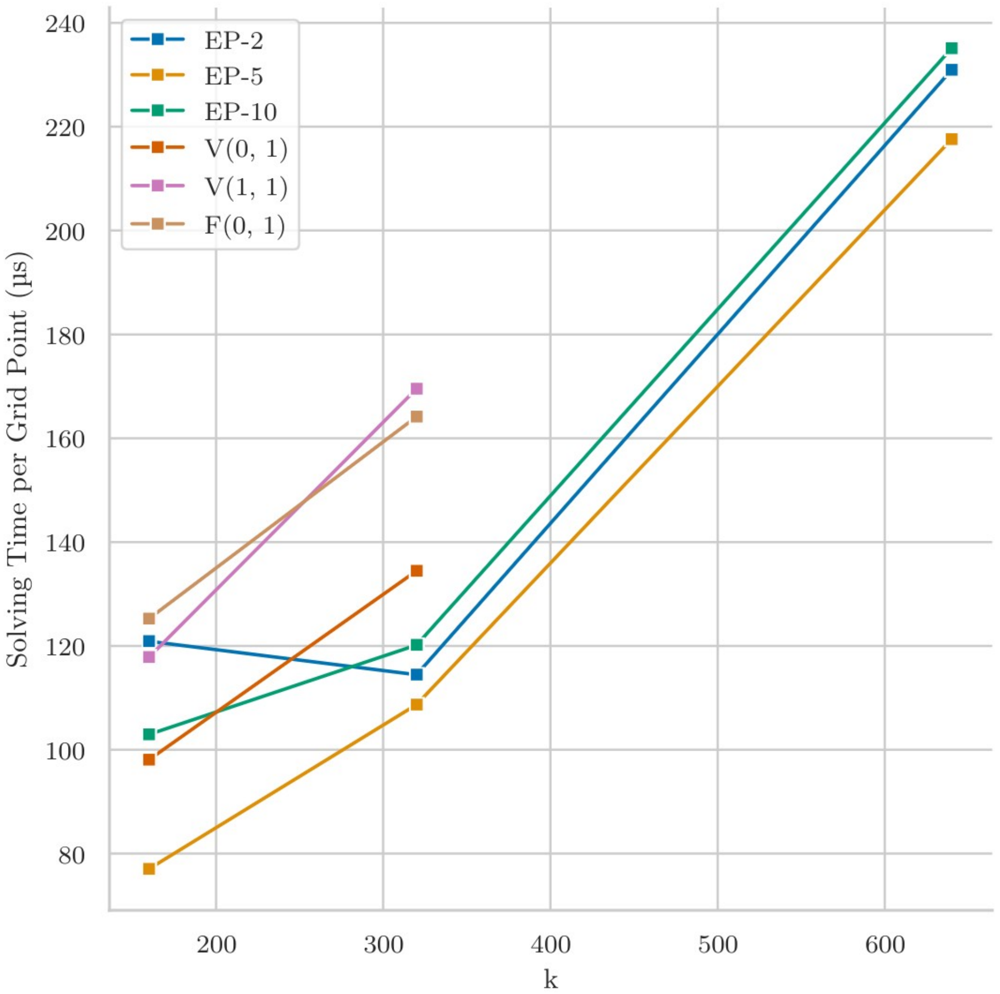
$$(-\nabla^2 - k^2)u = f \quad \text{in } (0,1)^2$$
$$u = 0 \quad \text{on } (0,1)\times\{0\}, (0,1)\times\{1\}$$
$$\partial_{\mathbf{n}} u - iku = 0 \quad \text{on } \{0\}\times(0,1), \{1\}\times(0,1)$$
$$f(x,y) = \delta(x-0.5, y-0.5),$$

2D indefinite Helmholtz problem.



(a) EP-5

Computational structure of the evolved multigrid preconditioners. The color of the node denotes the type of operation. Black:Coarse-grid solver, Blue: Pointwise Jacobi smoothing, Red: Red-black Gauss-Seidel smoothing, White: No operation. The relaxationfactor of each smoothing step is included in each node, while for coarse-grid correction, it is attached to the respective edge.



Solving time comparison of the best preconditioners.

References

[1] Schmitt, J., Kuckuk, S. & Köstler, H. EvoStencils: a grammar-based genetic programming approach for constructing efficient geometric multigrid methods. Genet Program Evolvable Mach 22, 511–537 (2021). https://doi.org/10.1007/s10710-021-09412-w

[2] Dinesh Parthasarathy, Jonas Schmitt, and Harald Köstler. 2023. Evolving Nonlinear Multigrid Methods With Grammar-Guided Genetic Programming. In Proceedings of the Companion Conference on Genetic and Evolutionary Computation (GECCO '23 Companion). Association for Computing Machinery, New York, NY, USA, 615–618. https://doi.org/10.1145/3583133.3590734

[3] Jonas Schmitt and Harald Köstler. 2022. Evolving generalizable multigrid-based helmholtz preconditioners with grammar-guided genetic programming. In Proceedings of the Genetic and Evolutionary Computation Conference (GECCO '22). Association for Computing Machinery, New York, NY, USA, 1009–1018. https://doi.org/10.1145/3512290.3528688

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